Section 20

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Let $a_1, \ldots, a_n, b$ be functions of $x_1, \ldots x_n, z$

To solve the first order differential equation

$$a_1 \frac{\partial z}{\partial x} + \ldots + a_n \frac{\partial z}{\partial x_n} = b, \tag{1}$$

It is necessary to solve the following symmetric system of first order differential equations

$$\frac{dx_1}{a_1} = \ldots = \frac{dx_n}{a_n} = \frac{dz}{b} \tag{2}$$

indeed to find n independent first integrals of this system:

$$\begin{aligned} \phi_1 (x_1, \ldots, x_n, z) &= C_1 \\ \phi_n (x_1, \ldots, x_n, z) &= C_n. \end{aligned} \tag{3}$$

If these integrals can be found, then every solution of system (1) can be written in the form

$$F(\phi_1, \ldots, \phi_n) = 0, \tag{4}$$

where F is an arbitrary differentiable function.

In particular, suppose z occurs explicitly in just one of the first integrals (3) for example, in the last one. Then

if F is an arbitrary differentiable function the general solution can be written in the form

$$\phi_n (x_1, \ldots, x_n, z) = f(\phi_1, \ldots, \phi_{n-1}). \tag{5}$$

If we then solve (5) for z, we obtain the general solution of (1) in explicit form.

Suppose we wish to find a surface $z = z(x,y)$ that satisfies the differential equation

$$a_1 (x, y, z)\frac{\partial z}{\partial x} + a_2 (x, y, z)\frac{\partial z}{\partial y} = b (x, y, z) \tag{6}$$

and that goes through a given curve

$$x = u(t), \quad y = v(t), z = w(t). \tag{7}$$

The following procedure will be successful. We first find two independent first integrals of the system

$$\frac{dx}{a_1} = \frac{dy}{a_2} = \frac{dz}{b}. \tag{8}$$

Suppose these are

$$\phi_1 (x, y, z) = C_1, \quad \phi_2(x, y, z) = C_2 \tag{9}$$

In these latter equations we replace x,y,z by their values from equation (7) and obtain in this way two equations in the parameter t:

$$\vartheta_1 (t) = C_1, \quad \vartheta_2 (t) = C_2 \tag{10}$$

We now eliminate t and obtain a relation of the following form:

$$F(C_1, C_2) = 0.$$

Finally we replace C_1, C_2 by the functions ϕ_1, ϕ_2 from the left members of the integrals (9), and have a surface of the required type.

If it happens that the two equations (10) are independent of t, it must be true that the curve (7) lies on every solution of the system (6), that is, it is a characteristic of this system. In this case the Cauchy problem has infinitely many solutions.

Example. Find the general solution of the equation

$$xz\,\partial z/\partial x + yz\,\partial z/\partial y = -xy \tag{11}$$

and find a surface which contains the curve

$$y = x^2,\ z = x^3 \tag{12}$$

Solution: The system of equations to be solved is

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{-xy}$$

for which two first integrals were found in section 19:

$$\frac{x}{y} = C_1,\ z^2 + xy = C_2 \tag{13}$$

Therefore, the general solution of (11) can be written in the form

$$F\left(\frac{x}{y},\ z^2 + xy\right) = 0,$$

where F is an arbitrary function. Since z is absent from one of the first integrals, the general solution can also be written in explicit form as follows:

$$z^2 + xy = f\left(\frac{x}{y}\right);\quad z = \pm\sqrt{f\left(\frac{x}{y}\right) - xy}$$

Here f is an arbitrary function.

To find the integral surface which contains the curve (12) we must first write the equations of this curve in parametric form:

$$x = x,\quad y = x^2,\quad z = x^3.$$

We now substitute this in (13) and obtain

$$1/x = C_1,\quad x^6 + x^3 = C_2\ .$$

x can be eliminated as follows:

$$\frac{1}{C_1^6} + \frac{1}{C_1^3} = C_2.$$

In this latter equation C_1 and C_2 stand for expressions in the system (13) and thus the integral surface required is:

$$\left(\frac{y}{x}\right)^6 + \left(\frac{y}{x}\right)^3 = z^2 + xy.$$

Find the general solution for problems 946-963.

946. $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=x-y$. **947.** $e^x\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=ye^x$.

948. $2x\frac{\partial z}{\partial x}+(y-x)\frac{\partial z}{\partial y}-x^2=0$.

949. $xy\frac{\partial z}{\partial x}-x^2\frac{\partial z}{\partial y}=yz$.

950. $x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial y}=x^2y+z$.

951. $(x^2+y^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}+z^2=0$.

952. $2y^4\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}=x\sqrt{z^2+1}$.

953. $x^2z\frac{\partial z}{\partial x}+y^2z\frac{\partial z}{\partial y}=x+y$.

954. $yz\frac{\partial z}{\partial x}-xz\frac{\partial z}{\partial y}=e^z$.

955. $(z-y)^2\frac{\partial z}{\partial x}+xz\frac{\partial z}{\partial y}=xy$.

956. $xy\frac{\partial z}{\partial x}+(x-2z)\frac{\partial z}{\partial y}=yz$.

957. $y\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial y}=\frac{y}{x}$.

958. $\sin^2 x\frac{\partial z}{\partial x}+\operatorname{tg} z\frac{\partial z}{\partial y}=\cos^2 z$.

959. $(x+z)\frac{\partial z}{\partial x}+(y+z)\frac{\partial z}{\partial y}=x+y$.

960. $(xz+y)\frac{\partial z}{\partial x}+(x+yz)\frac{\partial z}{\partial y}=1-z^2$.

961. $(y+z)\frac{\partial u}{\partial x}+(z+x)\frac{\partial u}{\partial y}+(x+y)\frac{\partial u}{\partial z}=u$.

962. $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+(z+u)\frac{\partial u}{\partial z}=xy$.

963. $(u-x)\frac{\partial u}{\partial x}+(u-y)\frac{\partial u}{\partial y}-z\frac{\partial u}{\partial z}=x+y$.

In problems 964-980 find the surface which satisfies the differential equation in question and contains the given curve.

964. $y^2\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=x;\quad x=0,\quad z=y^2.$

965. $x\frac{\partial z}{\partial x}-2y\frac{\partial z}{\partial y}=x^2+y^2;\quad y=1,\quad z=x^2.$

966. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy;\quad x=2,\quad z=y^2+1.$

967. $\operatorname{tg} x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z;\quad y=x,\quad z=x^3.$

968. $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=z^2(x-3y);\quad x=1,\quad yz+1=0.$

969. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-x^2-y^2;\quad y=-2,\quad z=x-x^2.$

970. $yz\frac{\partial z}{\partial x}+xz\frac{\partial z}{\partial y}=xy;\quad x=a,\quad y^2+z^2=a^2.$

971. $z\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}=2xz;\quad x+y=2,\quad yz=1.$

972. $z\frac{\partial z}{\partial x}+(z^2-x^2)\frac{\partial z}{\partial y}+x=0;\quad y=x^2,\quad z=2x.$

973. $(y-z)\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y;\quad z=y=-x.$

974. $x\frac{\partial z}{\partial x}+(xz+y)\frac{\partial z}{\partial y}=z;\quad x+y=2z,\quad xz=1.$

975. $y^2\frac{\partial z}{\partial x}+yz\frac{\partial z}{\partial y}+z^2=0;\quad x-y=0,\quad x-yz=1.$

976. $x\frac{\partial z}{\partial x}+z\frac{\partial z}{\partial y}=y;\quad y=2z,\quad x+2y=z.$

977. $(y+2z^2)\frac{\partial z}{\partial x}-2x^2z\frac{\partial z}{\partial y}=x^2;\quad x=z,\quad y=x^2.$

978. $(x-z)\frac{\partial z}{\partial x}+(y-z)\frac{\partial z}{\partial y}=2z;\quad x-y=2,\quad z+2x=1.$

979. $xy^3\frac{\partial z}{\partial x}+x^2z^2\frac{\partial z}{\partial y}=y^3z;\quad x=-z^3,\quad y=z^2.$

980*. $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2xy;\quad y=x,\quad z=x^2.$

981. Find the general equation of the surfaces which intersect the surfaces of the family $z^2 = Cxy$ orthogonally.

982. Find the equation of a surface which contains the line

$$y = x,\ z = 1$$

and is orthogonal to the surfaces

$$x^2 + y^2 + z^2 = Cx.$$

983. Find the partial differential equation that is satisfied by cylindrical surfaces with elements parallel to the vector [1,1,2]. State the general solution of tnis equation.

984. Use the result of the preceding problem to find the equation of a cylindrical surface with elements parallel to the vector [1,1,2], given that its generating curve is

$$x + y + z = 0,\ 5x^2 + 6xy + 5y^2 = 4.$$

985. Set up and solve the partial differential equation that is satisfied by all conical surfaces with vertex in a fixed point (a,b,c).

986. Find the equation of the family of surfaces with the following property: The abscissa of the point of intersection of an arbitrary tangent plane with the x-axis is half as great as the abscissa of the point of tangency.

In problems 987-989 solve the given system of equations.

987. $\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{z}{x}, \\ \dfrac{\partial z}{\partial y} = \dfrac{2z}{y}. \end{cases}$

988. $\begin{cases} \dfrac{\partial z}{\partial x} = y - z, \\ \dfrac{\partial z}{\partial y} = xz. \end{cases}$

989. $\begin{cases} \dfrac{\partial z}{\partial x} = 2yz - z^2, \\ \dfrac{\partial z}{\partial y} = xz. \end{cases}$

The problem of Pfaff is to be solved by rewriting the differential equation so that it becomes exact. This leads to partial differential equations for the intergrating factor; the latter are to be solved by the methods already explained.

In problems 990-993 find the surface which satisfies the given Pfaffian condition.

990. $(x - y)\,dx + z\,dy - x\,dz = 0.$
991. $3yz\,dx + 2xz\,dy + xy\,dz = 0.$
992. $(z + xy)\,dx - (z + y^2)\,dy + y\,dz = 0.$
993. $(2yz + 3x)\,dx + xz\,dy + xy\,dz = 0.$

ANSWERS

Note. In all places, even where not so indicated, the arguments of logarithms are absolute values. The answer $x = C_1 \exp y$ is sometimes written $y = \mathrm{LN}\ Cx$, without any requirement that C and x have the same sign. The precise form would be $y = \mathrm{LN}\ |Cx|$. In some cases, printing difficulties have required the omission of the absolute value sign, but it must be understood when needed. Only when the argument of the logarithm is necessarily positive for all allowable real values of the variables can the symbol be omitted.

1. Isoclines, Construction of the Differential Equation for a Family of Curves Isogonal Trajectories

15. $f(x, y) = 0$; $f'_x < 0$ (max), $f'_x > 0$ (min). 16. $f'_x + f \cdot f'_y = 0$. 17. $y = e^{\frac{xy'}{y}}$. 18. $y' = 3y^{\frac{2}{3}}$. 19. $xy' = 3y$. 20. $y^2 + y'^2 = 1$. 21. $x^2y' - xy = yy'$. 22. $2xyy' - y^2 = 2x^3$. 23. $y'^3 = 4y(xy' - 2y)$. 24. $y' = \cos \frac{x\sqrt{1 - y'^2}}{y}$. 25. $x(x-2)y'' - (x^2-2)y' + 2(x-1)y = 0$. 26. $(yy'' + y'^2)^2 = -y^3y''$. 27. $(1 - x \operatorname{ctg} x)y'' - xy' + y = 0$. 28. $x^3y''' - 3x^2y'' + 6xy' - 6y - 0$. 29. $y'''y' = 3y''^2$. 30. $(y - 2x)^2(y'^2 + 1) = (2y'^2 + 1)^2$. 31. $xy'^2 = y(2y' - 1)$. 32. $[x - y(\sqrt{2} + 1)]^2(y'^2 + 1) = (x + yy')^2$. 33. $x^2y'' - 2xy' + 2y = 0$. 34. $(y''y + y'^2 + 1)^2 = (y'^2 + 1)^3$. 35. $yy' + zz' = 0$, $y^2 + 2xzz' = x^2z'^2$. 36. $x^2 + y^2 = z^2 - 2z(y - xy')$; $x + yy' = zz' - z'(y - xy')$. 37. $4yy' = -x$. 38. $y' = -2y$. 39. $(x^2 + y)y' = -x$. 40. $(x + y)y' = y - x$; $(x - y)y' = x + y$. 41. $(x \mp y\sqrt{3})y' = y \pm x\sqrt{3}$. 42. $(3x \mp y\sqrt{3})y' = y \pm 3x\sqrt{3}$. 43. $(2x \mp y\sqrt{3})y' = y \pm 2x\sqrt{3}$. 44. $r' \sin \theta = r^2$. 45. $r' = \frac{1}{2} r \operatorname{ctg} \theta$. 46. $r' = r \operatorname{ctg} (\theta \pm 45°)$. 47. $(x+2y)y' = -3x - y$; $(3x + 2y)y' = y - x$. 48. $y'[2xy \pm (x^2 - y^2)] = y^2 - x^2 \pm 2xy$. 49. $x(1 + y'^2) = -2yy'$. 50. $yy'^3 + xy'^2 = -1$.

19. Non-Linear Systems.

921. $y = C_2 e^{C_1 x^2}$, $z = \frac{1}{2C_1 C_2} e^{-C_1 x^2}$. 922. $y = C_2 e^{C_1 x}$, $z = x + \frac{C_2}{C_1} e^{C_1 x}$. 923. $y = \frac{x + C_1}{x + C_2}$, $z = \frac{(C_2 - C_1)\, x}{(x + C_2)^2}$. 924. $y = C_2 e^{C_1 x^2}$, $z = \frac{2C_1}{C_2} x e^{-C_1 x^2}$. 925. $y = -\frac{1}{C_1} + \frac{C_1}{2}(x + C_2) - \frac{C_1}{4}(x + C_2)^2$, $z = \frac{C_1}{4}(x + C_2)^2 + \frac{1}{C_1}$. 926. $y = C_1 z$, $x = 2y - z + C_2$. 927. $x^2 - y^2 = C_1$, $x + y = C_2 z$. 928. $x - y = C_1 (y - z)$, $(x + y + z)(x - y)^2 = C_2$. 929. $x + z = C_1$, $(x + y + z)(y - 3x - z) = C_2$. 930. $x^2 - z^2 = C_1$, $y^2 - u^2 = C_2$, $(x + z) = C_3 (u + y)$. 931. $x + z = C_1$, $y + u = C_2$. $(x - z)^2 + (y - u)^2 = C_3$. 932. $x^2 - 2y = C_1$, $6xy - 2x^3 - 3z^2 = C_2$. 933. $y^2 + z^2 = C_1$, $x - yz = C_2$. 934. $x = C_1 y$, $xy - z = C_2 x$. 935 $x = C_1 y$, $xy - 2\sqrt{z^2 + 1} = C_2$. 936. $y = C_1 z$, $x - y^2 - z^2 = C_2 z$. 937. $y^2 + z^2 = C_1$, $x(y - z) = C_2$. 938. $xz = C_1$, $xy + z^2 = C_2$. 939. $x + z - y = C_1$, $\ln|x| + \frac{z}{y} = C_2$. 940. $x^2 + y^2 + z^2 = C_1$, $yz = C_2 x$. 941. 1: yes; 2: no. 942. 1: no; 2: yes.

943. Yes. 944. Dependent.

20. First Order Partial Differential Equations.

946. $F(x^2 - y^2, x - y + z) = 0$. 947. $F\left(e^{-x} - y^{-1}, z + \frac{x - \ln|y|}{e^{-x} - y^{-1}}\right) = 0$. 948. $F\left(x^2 - 4z, \frac{(x + y)^2}{x}\right) = 0$. 949. $F\left(x^2 + y^2, \frac{z}{x}\right) = 0$. 950. $F\left(\frac{x^2}{y}, xy - \frac{3z}{x}\right) = 0$. 951. $F\left(\frac{1}{x + y} + \frac{1}{z}, \frac{1}{x - y} + \frac{1}{z}\right) = 0$. 952. $F(x^2 + y^4, y(z + \sqrt{z^2 + 1})) = 0$. 953. $F\left(\frac{1}{x} - \frac{1}{y}, \ln|xy| - \frac{z^2}{2}\right) = 0$. 954. $F\left(x^2 + y^2, \operatorname{arctg}\frac{x}{y} + (z + 1)\, e^{-z}\right) = 0$. 955. $F(z^2 - y^2, x^2 + (y - z)^2) = 0$. 956. $F\left(\frac{z}{x}, 2x - 4z - y^2\right) = 0$. 957. $F(z - \ln|x|, 2x(z - 1) - y^2) = 0$. 958. $F(\operatorname{tg} z + \operatorname{ctg} x, 2y + 2\operatorname{tg} z \cdot \operatorname{ctg} x + \operatorname{ctg}^2 x) = 0$. 959. $F\left(\frac{x + y + z}{(x - y)^2}, (x - y)(x + y - 2z)\right) = 0$. 960. $F((x - y)(z + 1), (x + y)(z - 1)) = 0$. 961. $F\left(u(x - y), u(y - z), \frac{x + y + z}{u^2}\right) =$ 962. $F\left(\frac{x}{y}, xy - 2u, \frac{z + u - xy}{x}\right) = 0$. 963. $F\left(\frac{x - y}{z}, (2u + x + y)\, z, \frac{u - x - y}{z^2}\right) = 0$. 964. $y^2 - x^2 - \ln\sqrt{y^2 - x^2} = z - \ln|y|$. 965. $2x^2(y + 1) = y^2 + 4z - 1$. 966. $(x + 2y)^2 = 2x(z + xy)$.

967. $\sqrt{\frac{z}{y^3}}\sin x = \sin\sqrt{\frac{z}{y}}$. 968. $2xy + 1 = x + 3y + z^{-1}$.
969. $x - 2y = x^2 + y^2 + z$. 970. $2x^2 - y^2 - z^2 = a^2$.
971. $[(y^2z - 2)^2 - x^2 + z]\, y^2z = 1$. 972. $x^2 + z^2 = 5(xz - y)$.
973. $3(x + y + z)^2 = x^2 + y^2 + z^2$. 974. $xz = (xz - y - x + 2z)^2$.
975. $(1 + yz)^3 = 3yz(1 + yz - x) + y^3$. 976. $x + y + z = 0$.
977. $2(x^3 - 4z^3 - 3yz)^2 = 9(y + z^2)^3$. 978. $(x - y)(3x + y + 4z) = 4z$.

979. $xz + y^2 = 0$. 980. $z = xy + f(y/x)$, where f is any differentiable function satisfying $f(1) = 0$.

981. $F(x^2 - y^2, 2x^2 + z^2) = 0$. 981. $F(x^2 - y^2, 2x^2 + z^2) = 0$.

983. $F(2x - z, x - y) = 0$. 984. $(x - y)^2 + (z - x - y)^2 = 4$.

985. $F\left(\frac{y - b}{x - a}, \frac{z - c}{x - a}\right) = 0$. 986. $F\left(\frac{x^2}{y}, \frac{z}{y}\right) = 0$. 987. $z = Cxy^2$.

988. No solution. 989. $z = 0$. 990. No solution. 991. $x^3y^2z = C$.

992. $z = y^2 - xy$. 993. $x^2yz = C - x^3$; $x = 0$.